FINAL FORMULAS

Useful formulas

1. LENGTHS, AREAS, AND VOLUMES
   - The circumference of a circle with radius $R$ is
     \[ L = 2\pi R. \]
   - The area of a disk with radius $R$ is
     \[ \pi R^2. \]
   - The volume of a ball of radius $R$ is
     \[ V = \frac{4}{3}\pi R^3. \]
   - The surface area of a sphere with radius $R$ is
     \[ A = 4\pi R^2. \]
   - The volume of a cone with height $h$ and whose base has radius $R$ is
     \[ \frac{1}{3}\pi R^2h. \]
   - The surface area of a cone with height $h$ and whose base has radius $R$ is
     \[ \pi R(R + \sqrt{h^2 + R^2}). \]

2. INTEGRATION OF FUNCTIONS OF MORE THAN ONE VARIABLE
   - The \textbf{double integral}
     \[ \int_R f(x, y) \, dA = \int_R f(x, y) \, dx \, dy \]
     is equal to the signed volume of the 3-dimensional region between the graph of $f$ and the $xy$-plane over the region $R$.
   - The \textbf{triple integral}
     \[ \int_R f(x, y, z) \, dV = \int_R f(x, y, z) \, dx \, dy \, dz \]
     is equal to the average value of $f$ times the volume of the 3-dimensional region $R$. 
3. Change of co-ordinates

- **Polar co-ordinates** on a plane are given by

\[ x = r \cos \theta \quad \text{and} \quad y = r \sin \theta. \]

The area form for integrating a double integral is given by

\[ dA = dx \, dy = r \, dr \, d\theta. \]

- **Cylindrical co-ordinates** in 3-dimensional space are given by

\[
\begin{align*}
  x &= r \cos \theta \\
  y &= r \sin \theta \\
  z &= z,
\end{align*}
\]

where

\[ r \geq 0, \quad 0 \leq \theta < 2\pi, \quad \text{and} \quad -\infty < z < \infty. \]

The volume form for integrating a triple integral is given by

\[ dV = dx \, dy \, dz = r \, dr \, d\theta \, dz. \]

- **Spherical co-ordinates** in 3-dimensional space are given by

\[
\begin{align*}
  x &= \rho \sin \phi \cos \theta \\
  y &= \rho \sin \phi \sin \theta \\
  z &= \rho \cos \phi,
\end{align*}
\]

where

\[ \rho \geq 0, \quad 0 \leq \phi \leq \pi, \quad \text{and} \quad 0 \leq \theta < 2\pi. \]

The volume form for integrating a triple integral is given by

\[ dV = dx \, dy \, dz = \rho^2 \sin \phi \, d\rho \, d\phi \, d\theta. \]

4. Derivatives

- The **gradient** of a function \( f \) is the vector field

\[ \nabla f = f_x \vec{i} + f_y \vec{j} + f_z \vec{k}. \]

- The **curl** of a 2-dimensional vector field \( \vec{F} = F_1 \vec{i} + F_2 \vec{j} \) is the function

\[ \nabla \times \vec{F} = \partial_x F_2 - \partial_y F_1. \]

- The **curl** of a 3-dimensional vector field \( \vec{F} = F_1 \vec{i} + F_2 \vec{j} + F_3 \vec{k} \) is the vector field

\[ \nabla \times \vec{F} = i(\partial_y F_3 - \partial_z F_2) + j(\partial_z F_1 - \partial_x F_3) + k(\partial_x F_2 - \partial_y F_1). \]

- The **divergence** of a 3-dimensional vector field \( \vec{F} = F_1 \vec{i} + F_2 \vec{j} + F_3 \vec{k} \) is the function

\[ \nabla \cdot \vec{F} = \partial_x F_1 + \partial_y F_2 + \partial_z F_3. \]
5. Path-independent vector fields

- A vector field \( \vec{F} \) is a **gradient or path-independent vector field** on a region \( R \), if there is a function \( f \) such that \( \vec{F} = \vec{\nabla} f \).

- If a 2-dimensional vector field \( \vec{F} = F_1 \vec{i} + F_2 \vec{j} \) is path-independent, then
  \[
  \vec{\nabla} \times \vec{F} = \partial_x F_2 - \partial_y F_1 = 0.
  \]

- If a 3-dimensional vector field \( \vec{F} \) path-independent, then
  \[
  \vec{\nabla} \times \vec{F} = \vec{0}.
  \]

6. Line integrals

- Given a vector field \( \vec{F}(x, y, z) \) and a curve \( C \) parameterized by
  \[
  \vec{r}(t) = (x(t), y(t), z(t)),
  \]
  where \( \vec{r}(a) \) is the starting point of the curve and \( \vec{r}(b) \) the ending point, the **line integral** of \( \vec{F} \) along the curve \( C \) is given by
  \[
  \int_C \vec{F} \cdot d\vec{r} = \int_a^b \vec{F}(\vec{r}(t)) \cdot \vec{r}'(t) \, dt.
  \]

- If \( \vec{F} = F_1 \vec{i} + F_2 \vec{j} + F_3 \vec{k} \), then the line integral above can also be written as:
  \[
  \int_C F_1 \, dx + F_2 \, dy + F_3 \, dz = \int_a^b F_1 x' + F_2 y' + F_3 z' \, dt.
  \]

- **Fundamental Theorem of Line Integrals.** If \( \vec{F} \) is a gradient or path-independent vector field on a region \( R \) and is given by
  \[
  \vec{F} = \vec{\nabla} f,
  \]
  then given any curve \( C \) in \( R \) that starts at a point \( P \) and ends at a point \( Q \), the line integral of \( \vec{F} \) along \( C \) is given by
  \[
  \int_C \vec{F} \cdot d\vec{r} = f(Q) - f(P),
  \]
  where \( \vec{F} = \vec{\nabla} f \).

- In particular, if \( \vec{F} \) is a gradient vector field and \( C \) is a closed curve, then
  \[
  \oint_C \vec{F} \cdot d\vec{r} = 0.
  \]

7. Green’s theorem

Suppose \( C \) is a closed oriented curve that is the boundary of a 2-dimensional region \( R \), which lies to the left of \( C \). Then given a vector field \( \vec{F} = F_1 \vec{i} + F_2 \vec{j} \) on \( R \),
\[
\oint_C \vec{F} \cdot d\vec{r} = \int_R \partial_x F_2 - \partial_y F_1 \, dx \, dy.
\]
8. Flux integral

- An **orientation of a surface** \( S \) in 3-dimensional space is a consistent continuous choice of the direction of the normal vector at every point in \( S \).
- The **flux integral of a vector field** \( \vec{F} \) across an oriented surface \( S \) in 3-dimensional space is written as
  \[
  \int_S \vec{F} \cdot d\vec{A} = \int_S \vec{F} \cdot \vec{n} \, dA,
  \]
  where \( \vec{n} \) is the unit normal pointing in the direction specified by the orientation of \( S \).

9. Flux integral across the side of a cylinder

- **Cylindrical coordinates** \( r, \theta, \) and \( z \) satisfy
  \[
  x = r \cos \theta, \quad y = r \sin \theta, \quad z = z,
  \]
  where \( r \geq 0 \) and \( 0 \leq \theta \leq 2\pi \).
- If \( S \) is the side of a cylinder with radius \( R \), then the outer unit normal is given by
  \[
  \vec{n} = \frac{x\vec{i} + y\vec{j}}{\sqrt{x^2 + y^2}} = \vec{i} \cos \theta + \vec{j} \sin \theta.
  \]
- The **flux of a vector field** \( \vec{F} \) across all or a piece of the side of the cylinder \( S \) with radius \( R \), centered on the \( z \)-axis, and oriented away from the \( z \)-axis is given by
  \[
  \int_S \vec{F} \cdot d\vec{A} = \int_S \vec{F} \cdot \vec{n} \, dA = \int_S \vec{F} \cdot ((\cos \theta)\vec{i} + (\sin \theta)\vec{j})R \, dz \, d\theta.
  \]

10. Flux integral across a sphere

- **Spherical coordinates** \( \rho, \theta, \phi \) satisfy
  \[
  x = \rho \cos \theta \sin \phi, \quad y = \rho \sin \theta \sin \phi, \quad z = \cos \phi,
  \]
  where \( \rho \geq 0 \), \( 0 \leq \theta \leq 2\pi \), and \( 0 \leq \phi \leq \pi \).
- The outer unit normal of a sphere of radius \( R \) is given by
  \[
  \vec{n} = \frac{\vec{r}}{|\vec{r}|} = \frac{x\vec{i} + y\vec{j} + z\vec{k}}{\sqrt{x^2 + y^2 + z^2}} = \vec{i} \cos \theta \sin \phi + \vec{j} \sin \theta \sin \phi + \vec{k} \cos \phi.
  \]
- The **flux of a vector field** \( \vec{F} \) across a surface \( S \) lying on the surface of the sphere of radius \( R \) centered at the origin and oriented outward is given by
  \[
  \int_S \vec{F} \cdot d\vec{A} = \int_S \vec{F} \cdot \vec{n} \, dA = \int_S \vec{F} \cdot ((\sin \phi \cos \theta)\vec{i} + (\sin \phi \sin \theta)\vec{j} + (\cos \phi)\vec{k})R^2 \, \sin \phi \, d\phi \, d\theta.
  \]
11. Stokes’ Theorem

Given an oriented surface $S$ 3-dimensional space with boundary $C$ oriented so that $S$ lies to the left of $S$ and a vector field $\vec{F}$ defined on a region containing $S$,

$$\int_C \vec{F} \cdot d\vec{r} = \int_S (\nabla \times \vec{F}) \cdot d\vec{A}.$$ 

12. Divergence Theorem

Given a 3-dimensional region $R$ with boundary $S$ oriented outward and a vector field $\vec{F}$ defined on a domain containing $R$,

$$\int_S \vec{F} \cdot d\vec{A} = \int_R \nabla \cdot \vec{F} dV.$$